Summary of definitions, concepts and relationships for time series

1. A random variable X is bounded if it holds that $|X| \leq M$ for some $M \in \mathbb{R}$.

2. A random variable X is said to be integrable if it holds that $\mathsf{E}|X| < \infty$.

3. A random variable X is said to be squared integrable if it holds that $\mathsf{E}|X|^2 < \infty$, i.e. if X has finite second moments.

4. A time series, or stochastic process, is a collection of random variables $\{X_t\}_{t\in\mathbb{Z}}$ indexed by time.

5. Concepts related to boundedness and integrability in the context of a time series is *uniform boundedness* and *uniform integrability.* These concepts are useful when imposing structure on the time series.

6. A time series or stochastic process is a collection of random variables $\{X\}_{t\in\mathbb{Z}}$ indexed by time.

7. The autocovariance function of a time series is given by the covariance between two random variables of the time series, i.e.

$$\gamma(t, s) \stackrel{\text{def}}{=} \mathsf{Cov}\left[X_t, X_s\right] \tag{1}$$

8. A time series is said to be strictly stationary if the joint probability distribution of any two finite stretches are identical, i.e.

$$\mathsf{P}\left(X_t, X_{t+1}, X_{t+2}, \ldots, X_{t+k}\right) = \mathsf{P}\left(X_s, X_{s+1}, X_{s+2}, \ldots, X_{s+k}\right) \tag{2}$$

   for any indices $t, s \in \mathbb{Z}$ and all $k \in \mathbb{N}$. Note that this definition implies that the marginal distributions of the individual random variables of the time series are identical.

9. A time series is said to be covariance or wide sense stationary if it has finite constant mean and its autocovariance function does not depend on the time indices t and s but

only on the distance between them $h = |t - s|$. That is it holds that $\mathsf{E}[X_t] = \mu$ for all t and

$$\gamma(h) = \mathsf{E}[X_t X_{t-h}] - \mu^2 \tag{3}$$

Note that this definition implies that $\mathsf{Var}[X_t] = \gamma(0)$ constant for all t and one can define the autocorrelation function as

$$\rho(h) \stackrel{\text{def}}{=} \frac{\gamma(h)}{\gamma(0)} \tag{4}$$

10. A strictly stationary time series with finite second moments is also covariance stationary; the converse is not necessarily true.

11. For a covariance stationary time series we have that $\lim_{h\to\infty} \rho(h) = 0$.

12. A covariance stationary time series is said to have short memory if the the autocovariance function decays to zero exponentially fast and the sum of all autocovariance tends to zero, i.e.

$$\lim_{h\to\infty} \rho(h) \approx c^h \text{ and } \lim_{h\to\infty} \sum_{j=1}^{h} \rho(h) = 0 \tag{5}$$

for some constant c.

13. A covariance stationary time series is said to have long memory if the the autocovariance function decays to zero geometrically fast and the autocovariances are not summable, i.e.

$$\lim_{h\to\infty} \rho(h) \approx h^d \text{ and } \lim_{h\to\infty} \sum_{j=1}^{h} \rho(h) = \infty \tag{6}$$

for some constant d.

14. A strictly stationary time series is said to be ergodic if it satisfies certain conditions that allow the average of a single realization to converge to the mean of the common marginal distribution, i.e.

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} X_t = \mathsf{E}[X_1] \tag{7}$$

A similar result holds for a covariance stationary series.

15. A time series is said to satisfy a mixing condition if, under certain conditions, two collections of random variables of the time series are independent if they are sufficiently apart in time.

16. A time series of squared integrable random variables that satisfies any valid mixing condition is also ergodic. Note that the crucial requirement here is the existence of second moments - mixing alone is not enough for achieving ergodicity. Equivalently, a stationary time series that is mixing is also ergodic.